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STUDY OF SUBCLASS OF ANALYTIC FUNCTIONS IN THE UNIT DISK

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Abstract

In this paper we have introduce a new class $S^{**}(\alpha, \beta, \mu, a, c)$ of analytic functions involving C-S operator.This class generalises the class of uniformly starlike and convex functions. We investigate several properties of the class like coefficient inequality, growth and distortion theorem and extreme points. We also derive subordination results and integral mean inequalities for the class.

1. Introduction and Motivation

Let S denote the class of the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0, n \in N \tag{1.1}$$

which are analytic and univalent in the open unit disc $U = \{z : |z| < 1\}$. We also denote by C the class of functions $f(z) \in S$ that are convex in D

For two functions $f(z)$ and $g(z)$ given by

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$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad and$$
$$f(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

we define Hadamard product or convolution of f and g given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \qquad (z \in U) \tag{1.2}$$

Definition 1 : Let $g(z)$ be analytic and univalent in U. If $f(z)$ is analytic in $U, f(0) = g(0)$ and $f(U) \subset g(u)$, then we say that f is subordinate to g, and write $f \prec g$ or $f(z) \prec g(z)$. We also say that g is superordinate to f in U.

Definition 2 : An infinite sequence $\{d_n\}_{n=1}^{\infty}$ of complex number is called as subordinating factor sequence if for every univalent function f in C, the class of convex function in U, we have

$$\sum_{n=1}^{\infty} d_n a_n z^n \prec f(z) \qquad (z \in U; a_1 = 1)$$

Next we give a characterizing result of subordinating factor sequence in the form of a lemma due to Wilf [4].

Lemma 1.1 : The sequence $\{d_n\}_{n=1}^{\infty}$ is a subordinating factor if and only if,

$$Re\left\{1 + 2\sum_{n=1}^{\infty} d_n z^n\right\} > 0, \qquad (z \in U).$$

Let $\phi(a, c; z)$ be the incomplete beta function defined by

$$\phi(a, c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n \tag{1.3}$$

$a \in R;\ c \in R \backslash z_0^-; \qquad z_0^- = \{0, -1, -2, -3 \ldots\}, \qquad z \in U$

where, $(\lambda)_n$ is the Pochhammer Symbol defined in terms of the Gamma functions, by,

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma\lambda} = \begin{cases} 1 & , \quad n = 0 \\ \lambda(\lambda+1)(\lambda+2) \ldots\ldots (\lambda+n-1) & , \quad n \in N \end{cases}$$

corresponding to the function $\phi(a, c : z)$, Carlson and Shaffer [2] introduced a linear operator $L(a, c) : S \to S$ by

$$\begin{aligned} L(a,c)f(z) &= \phi(a,c:z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} a_n z^n \end{aligned} \tag{1.4}$$

where the operator $*$ stands for the hadamard product of the power series given by (1.2) Consider the function $\psi(z)$ defined by

$$\psi(z) = z + \sum_{n=2}^{\infty} \xi_n z^n \tag{1.5}$$

where $\xi_n \geq 0 \quad \text{for } n \in N \backslash \{1\}$
Let S^* be the subclass of S consisting of functions $f(z)$
Satisfying

$$Re\left\{1 - \frac{1}{\mu} + \frac{1}{\mu}\frac{z(L(a,c)f(z) * \psi(z))'}{L(a,c)f(z) * \psi(z)}\right\} > \beta \tag{1.6}$$

where $0 \leq \beta < 1, \quad \mu \neq 0, \quad L(a,c)f(z) * \psi(z) \neq 0$

Definition 3 : A function $f(z) \in S$ is said to be in the class of α - uniformly starlike function of order β denoted by $S_s(\alpha, \beta)$ if,

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha\left|\frac{zf'(z)}{f(z)} - 1\right| + \beta \tag{1.7}$$

for some $0 \leq \beta < 1, \quad \alpha \geq 0$ and $z \in U$

Definition 4 : A function $f(z) \in S$ is said to be in the class of α - uniformly convex function of order β denoted by $S_k(\alpha, \beta)$ if,

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha\left|\frac{zf''(z)}{f'(z)} - 1\right| + \beta \tag{1.8}$$

For some $0 \leq \beta < 1, \alpha \geq 0$, and $z \in U$

We introduce a new subclass $S^{**}(\alpha, \beta, \mu, a, c)$ of S consisting of function $f(z)$ which satisfy

$$Re\left\{1 - \frac{1}{\mu} + \frac{1}{\mu}.\frac{z(L(a,c)f(z) * \psi(z))'}{L(a,c)f(z) * \psi(z)}\right\} > \alpha\left|\frac{1}{\mu}.\frac{z(L(a,c)f(z) * \psi(z))'}{L(a,c)f(z) * \psi(z)} - 1\right| + \beta \tag{1.9}$$

Where $\mu \neq 0, \quad 0 \leq \beta < 1$

From (1.4) and (1.5), we have

$$L(a,c)f(z) * \psi(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} \xi_n a_n z^n \tag{1.10}$$

Remark 1 : By specializing the values $\mu = 1$ and $\psi(z) = \dfrac{z}{1-z}$, we obtain the class $S_s(\alpha, \beta)$. Also for $\mu = 1$ and $\psi(z) = \dfrac{z}{(1-z)^2}$, we obtain the class $S_s(\alpha, \beta)$.

This paper is motivated from the results of S. M. Khairnar and Meena More [1].

2. Main Result

In this section we obtain coefficient inequality for the class $S^{**}(\alpha, \beta, \mu, a, c)$.

Theorem 2.1 : If $f(z) \in S$ satisfies the given inequality,

$$\sum_{n=2}^{\infty} H(\alpha, \beta, \mu, a, c, \xi_n) |a_n| \leq 1 - \frac{1}{\mu} - \beta \tag{2.1}$$

where $H(\alpha, \beta, \mu, a, c, \xi_n) = \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} \left[\frac{1}{\mu}(1+\alpha)(n-1) + 1 - \frac{1}{\mu} - \beta \right] \xi_n$

$\alpha \geq 0, \quad 0 \leq \beta < 1, \quad \mu \neq 0, \quad L(a,c)f(z) * \psi(z) \neq 0$

then $f(z) \in S^{**}(\alpha, \beta, \mu, a, c, \xi_n)$

Proof : Suppose that the condition (2.1) holds for $\alpha \neq 0, \quad 0 \leq \beta < 1, \quad \mu \neq 0$ and $L(a,c)f(z) * \psi(z) \neq 0$

From (1.9), we have,

$$\alpha \left| \frac{1}{\mu} . \frac{z(L(a,c)f(z) * \psi(z))'}{L(a,c)f(z) * \psi(z)} - 1 \right| - Re \left\{ 1 - \frac{1}{\mu} + \frac{1}{\mu} \frac{z(L(a,c)f(z) * \psi(z))'}{L(a,c)f(z) * \psi(z)} \right\} \leq -\beta$$

Thus,

$$\alpha \left| \frac{1}{\mu} . \frac{z(L(a,c)f(z) * \psi(z))'}{L(a,c)f(z) * \psi(z)} - 1 \right| - Re \left\{ \frac{1}{\mu} . \frac{z(L(a,c)f(z) * \psi(z))'}{L(a,c)f(z) * \psi(z)} - 1 \right\} \leq 1 - \frac{1}{\mu} - \beta$$

Notice that,

$$\alpha \left| \frac{1}{\mu} . \frac{z(L(a,c)f(z) * \psi(z))'}{L(a,c)f(z) * \psi(z)} - 1 \right| - Re \left\{ \frac{1}{\mu} . \frac{z(L(a,c)f(z) * \psi(z))'}{L(a,c)f(z) * \psi(z)} - 1 \right\}$$

$$= (\alpha + 1) \left| \frac{1}{\mu} . \frac{z(L(a,c)f(z) * \psi(z))'}{L(a,c)f(z) * \psi(z)} - 1 \right|$$

$$\leq (\alpha+1)\frac{1}{\mu}.\frac{\sum_{n=2}^{\infty}(n-1)\frac{(a)_{n-1}}{(c)_{n-1}}\xi_n\,|a_n|}{1-\sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}\xi_n\,|a_n|}$$

The above inequality is bounded by $(1-\frac{1}{\mu}-\beta)$ if,

$$\sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}\left[\frac{1}{\mu}(1+\alpha)(n-1)+1-\frac{1}{\mu}-\beta\right]\xi_n\,|a_n| \leq 1-\frac{1}{\mu}-\beta$$

Hence the proof completes.

3. Growth and Distortion Theorem

Theorem 3.1 : Let $f(z) \in S^{**}(\alpha,\beta,\mu,a,c)$ and $\alpha \geq 0, \quad 0 \leq \beta < 1, \quad \mu \neq 0, \quad L(a,c)f(z) * \psi(z) \neq 0$ then,

$$(i) \qquad r - \frac{1-\frac{1}{\mu}-\beta}{\frac{(a)_{n-1}}{(c)_{n-1}}\left[\frac{1}{\mu}(1+\alpha)(n-1)+1-\frac{1}{\mu}-\beta\right]\xi_n}r^2$$

$$\leq |f(z)| \leq r + \frac{1-\frac{1}{\mu}-\beta}{\frac{(a)_{n-1}}{(c)_{n-1}}\left[\frac{1}{\mu}(1+\alpha)(n-1)+1-\frac{1}{\mu}-\beta\right]\xi_n}r^2, \quad |z| = r < 1$$

$$(ii) \qquad 1 - \frac{2(1-\frac{1}{\mu}-\beta)}{\frac{(a)_{n-1}}{(c)_{n-1}}\left[\frac{1}{\mu}(1+\alpha)(n-1)+1-\frac{1}{\mu}-\beta\right]\xi_n}r$$

$$\leq \left|f'(z)\right| \leq 1 + \frac{2(1-\frac{1}{\mu}-\beta)}{\frac{(a)_{n-1}}{(c)_{n-1}}\left[\frac{1}{\mu}(1+\alpha)(n-1)+1-\frac{1}{\mu}-\beta\right]\xi_n}r, \quad |z| = r < 1$$

The above results are sharp for

$$f(z) = z + \frac{1-\frac{1}{\mu}-\beta}{\frac{(a)_{n-1}}{(c)_{n-1}}\left[\frac{1}{\mu}(1+\alpha)(n-1)+1-\frac{1}{\mu}-\beta\right]\xi_n}z^2$$

Proof : Let $f(z) \in S^{**}(\alpha, \beta, \mu, a, c)$, we have

$$\sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} \left[\frac{1}{\mu}(1+\alpha)(n-1) + 1 - \frac{1}{\mu} - \beta\right] \xi_n |a_n| \leq 1 - \frac{1}{\mu} - \beta$$

This gives

$$\sum_{n=2}^{\infty} a_n \leq \frac{1 - \frac{1}{\mu} - \beta}{\frac{(a)_{n-1}}{(c)_{n-1}} \left[\frac{1}{\mu}(1+\alpha)(n-1) + 1 - \frac{1}{\mu} - \beta\right] \xi_n} \tag{3.1}$$

Therefore,

$$\begin{aligned} |f(z)| &\leq |z| + \sum_{n=2}^{\infty} a_n |z|^n \\ &\leq r + \sum_{n=2}^{\infty} a_n r^n, \quad |z| = r < 1 \\ &\leq r + r^2 \sum_{n=2}^{\infty} a_n \\ \text{using (3.1)} \Rightarrow &\leq r + \frac{1 - \frac{1}{\mu} - \beta}{\frac{(a)_{n-1}}{(c)_{n-1}} \left[\frac{1}{\mu}(1+\alpha)(n-1) + 1 - \frac{1}{\mu} - \beta\right] \xi_n} r^2, \end{aligned} \tag{3.2}$$

Similarly we have

$$\begin{aligned} |f(z)| &\geq |z| - \sum_{n=2}^{\infty} a_n |z|^n \\ &\geq r - \sum_{n=2}^{\infty} a_n r^n, \quad |z| = r < 1 \\ &\geq r - r^2 \sum_{n=2}^{\infty} a_n \\ \text{using (3.1)} \Rightarrow &\geq r - \frac{1 - \frac{1}{\mu} - \beta}{\frac{(a)_{n-1}}{(c)_{n-1}} \left[\frac{1}{\mu}(1+\alpha)(n-1) + 1 - \frac{1}{\mu} - \beta\right] \xi_n} r^2, \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3),we get

$$r - \frac{1-\frac{1}{\mu}-\beta}{\frac{(a)_{n-1}}{(c)_{n-1}}\left[\frac{1}{\mu}(1+\alpha)(n-1)+1-\frac{1}{\mu}-\beta\right]\xi_n} r^2 \leq |f(z)| \leq r + \frac{1-\frac{1}{\mu}-\beta}{\frac{(a)_{n-1}}{(c)_{n-1}}\left[\frac{1}{\mu}(1+\alpha)(n-1)+1-\frac{1}{\mu}-\beta\right]\xi_n} r^2$$

Thus we have proved the result (i)

Now since

$$f'(z) = 1+\sum_{n=2}^{\infty} n a_n z^{n-1}$$

$$wehave$$

$$\begin{aligned} |f'(z)| &\leq 1+\sum_{n=2}^{\infty} n a_n |z|^{n-1} \\ &\leq 1+\sum_{n=2}^{\infty} n a_n r^{n-1}, \quad |z| = r < 1 \qquad (3.4) \\ &\leq 1+r\sum_{n=2}^{\infty} a_n . r^{n-1} \end{aligned}$$

$$\text{using (3.1)} \Rightarrow \leq 1+\frac{2(1-\frac{1}{\mu}-\beta)}{\frac{(a)_{n-1}}{(c)_{n-1}}\left[\frac{1}{\mu}(1+\alpha)(n-1)+1-\frac{1}{\mu}-\beta\right]\xi_n} r,$$

Similarly we have

$$\begin{aligned}|f'(z)| &\geq 1-\sum_{n=2}^{\infty} n a_n |z|^{n-1} \\ &\geq 1-\sum_{n=2}^{\infty} n a_n r^{n-1}, \quad |z|=r<1 \\ &\geq 1-r\sum_{n=2}^{\infty} a_n r^{n-1} \qquad (3.5)\\ \text{using (3.1)} \Rightarrow &\geq 1-\frac{2(1-\frac{1}{\mu}-\beta)}{\frac{(a)_{n-1}}{(c)_{n-1}}\left[\frac{1}{\mu}(1+\alpha)(n-1)+1-\frac{1}{\mu}-\beta\right]\xi_n} r,\end{aligned}$$

Combining (3.4) and (3.5),we get

$$\begin{aligned}1 \quad - \quad &\frac{2(1-\frac{1}{\mu}-\beta)}{\frac{(a)_{n-1}}{(c)_{n-1}}\left[\frac{1}{\mu}(1+\alpha)(n-1)+1-\frac{1}{\mu}-\beta\right]\xi_n} r \leq |f'(z)| \\ \leq 1 \quad + \quad &\frac{2(1-\frac{1}{\mu}-\beta)}{\frac{(a)_{n-1}}{(c)_{n-1}}\left[\frac{1}{\mu}(1+\alpha)(n-1)+1-\frac{1}{\mu}-\beta\right]\xi_n} r\end{aligned}$$

Thus we have proved the result (ii)

The above results (i) and (ii) are sharp for the function

$$f(z)=z+\frac{1-\frac{1}{\mu}-\beta}{\frac{(a)_{n-1}}{(c)_{n-1}}\left[\frac{1}{\mu}(1+\alpha)(n-1)+1-\frac{1}{\mu}-\beta\right]\xi_n} z^2.$$

4. Extreme Points

In the following theorem we obtain Extreme Points of the class $S^{**}(\alpha,\beta,\mu,a,c)$.

Theorem 4.1 : Let $f_1(z)=z$ and

$$f_n(z)=z+\frac{1-\frac{1}{\mu}-\beta}{\frac{(a)_{n-1}}{(c)_{n-1}}\left[\frac{1}{\mu}(1+\alpha)(n-1)+1-\frac{1}{\mu}-\beta\right]\xi_n} z^2, \quad n\in N\backslash\{1\}$$

Then $f(z) \in S(\alpha,\beta,\mu,a,c)$ if and only if $f(z)$ can be expressed in the form
$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$ where $\lambda_n \geq 0$, $\sum_{n=1}^{\infty} \lambda_n = 1$

Proof : Suppose that

$$
\begin{aligned}
f(z) &= \sum_{n=1}^{\infty} \lambda_n f_n(z) \\
&= z + \sum_{n=2}^{\infty} \lambda_n \frac{1 - \frac{1}{\mu} - \beta}{\frac{(a)_{n-1}}{(c)_{n-1}} \left[\frac{1}{\mu}(1+\alpha)(n-1) + 1 - \frac{1}{\mu} - \beta\right] \xi_n} z^n \\
&= z + \sum_{n=2}^{\infty} e_n z^n, \quad where \\
e_n &= \frac{1 - \frac{1}{\mu} - \beta}{\frac{(a)_{n-1}}{(c)_{n-1}} \left[\frac{1}{\mu}(1+\alpha)(n-1) + 1 - \frac{1}{\mu} - \beta\right] \xi_n} \lambda_n
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} \left[\frac{1}{\mu}(1+\alpha)(n-1) + 1 - \frac{1}{\mu} - \beta\right] \xi_n e_n \\
&\sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} \left[\frac{1}{\mu}(1+\alpha)(n-1) + 1 - \frac{1}{\mu} - \beta\right] \xi_n \cdot \\
&\frac{1 - \frac{1}{\mu} - \beta}{\frac{(a)_{n-1}}{(c)_{n-1}} \left[\frac{1}{\mu}(1+\alpha)(n-1) + 1 - \frac{1}{\mu} - \beta\right] \xi_n} \\
= \ & 1 - \frac{1}{\mu} - \beta \sum_{n=2}^{\infty} \lambda_n \\
\leq \ & 1 - \frac{1}{\mu} - \beta, \quad \sum_{n=2}^{\infty} \lambda_n = 1
\end{aligned}
$$

Hence $f(z) \in S^{**}(\alpha,\beta,\mu,a,c)$

Conversly, suppose that $f(z) \in S^{**}(\alpha,\beta,\mu,a,c)$then using equation (2.1) we have

$$
a_n \leq \frac{1 - \frac{1}{\mu} - \beta}{\frac{(a)_{n-1}}{(c)_{n-1}} \left[\frac{1}{\mu}(1+\alpha)(n-1) + 1 - \frac{1}{\mu} - \beta\right] \xi_n}, n = 2, 3, \ldots
$$

Setting, $\lambda_n = \dfrac{\dfrac{(a)_{n-1}}{(c)_{n-1}}\left[\dfrac{1}{\mu}(1+\alpha)(n-1)+1-\dfrac{1}{\mu}-\beta\right]\xi_n}{1-\dfrac{1}{\mu}-\beta} a_n, \quad n = 2, 3, \ldots$

and $\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n$

We find that $\lambda_n \geq 0, \quad n = 1, 2, 3 \ldots.$ Thus $\lambda_n \geq 0, \quad n = 1, 2, 3 \ldots$ and $\sum_{n=2}^{\infty} \lambda_n = 1.$

$$\begin{aligned} \text{Now, } f(z) &= z + \sum_{n=2}^{\infty} a_n z^n \\ &= z + \sum_{n=2}^{\infty} \frac{1-\dfrac{1}{\mu}-\beta}{\dfrac{(a)_{n-1}}{(c)_{n-1}}\left[\dfrac{1}{\mu}(1+\alpha)(n-1)+1-\dfrac{1}{\mu}-\beta\right]\xi_n} \lambda_n z^n \\ &= \sum_{n=1}^{\infty} \lambda_n f_n(z) \end{aligned}$$

This completes the proof of the theorem.

5. Subordination Theorem and Integral Mean Inequalities

In this we obtain a sharp Subordination result associated with the class $S^{**}(\alpha, \beta, \mu, a, c)$. Also some interesting results of application of main results are investigated.

Theorem 5.1 : Let $f(z) \in S^{**}(\alpha, \beta, \mu, a, c)$ and $\{\xi_n\}_{n=2}^{\infty}$ be a non - decreasing sequence, then

$$\frac{H(\alpha, \beta, \mu, a, c, \xi_2)}{2\left\{H(\alpha, \beta, \mu, a, c, \xi_2) + 1 - \dfrac{1}{\mu} - \beta\right\}} (f * g)(z) \prec g(z) \tag{5.1}$$

for every function $g(z)$ in C, the class of convex function and

$$Re\{f(z)\} > -\frac{\left\{H(\alpha, \beta, \mu, a, c, \xi_2) + 1 - \dfrac{1}{\mu} - \beta\right\}}{H(\alpha, \beta, \mu, a, c, \xi_2)} \tag{5.2}$$

The constant factor

$$\frac{H(\alpha, \beta, \mu, a, c, \xi_2)}{2\left\{H(\alpha, \beta, \mu, a, c, \xi_2) + 1 - \dfrac{1}{\mu} - \beta\right\}} \tag{5.3}$$

cannot replace larger one.

Proof : Let $f(z) \in S^{**}(\alpha, \beta, \mu, a, c)$ and $g(z) = z + \sum_{n=2}^{\infty} d_n z^n$ be any function in the class C. Then we have

$$\frac{H(\alpha,\beta,\mu,a,c,\xi_2)}{2\left\{H(\alpha,\beta,\mu,a,c,\xi_2)+1-\frac{1}{\mu}-\beta\right\}}(f*g)(z)$$

$$= \frac{H(\alpha,\beta,\mu,a,c,\xi_2)}{2\left\{H(\alpha,\beta,\mu,a,c,\xi_2)+1-\frac{1}{\mu}-\beta\right\}}(z+\sum_{n=2}^{\infty} a_n d_n z^n) \tag{5.4}$$

Then by definition (1.1), the subordination result (5.1) will hold true if the sequence

$$\left\{\frac{H(\alpha,\beta,\mu,a,c,\xi_2)a_n}{2\left\{H(\alpha,\beta,\mu,a,c,\xi_2)+1-\frac{1}{\mu}-\beta\right\}}\right\}_{n=1}^{\infty} \tag{5.5}$$

is a subordinating factor sequence with $a_1 = 1$. In view of Lemma (1.1), this is equivalent to the following inequality

$$Re\left\{1+\sum_{n=1}^{\infty}\frac{H(\alpha,\beta,\mu,a,c,\xi_2)}{2\left\{H(\alpha,\beta,\mu,a,c,\xi_2)+1-\frac{1}{\mu}-\beta\right\}}a_n z^n\right\} > 0, \quad z \in U \tag{5.6}$$

In view of (2.1) and $|z| = r(0 < r < 1)$, we obtain

$$Re\left\{1+\sum_{n=1}^{\infty}\frac{H(\alpha,\beta,\mu,a,c,\xi_2)}{\left\{H(\alpha,\beta,\mu,a,c,\xi_2)+1-\frac{1}{\mu}-\beta\right\}}a_n z^n\right\} =$$

$$Re\left\{1+\frac{H(\alpha,\beta,\mu,a,c,\xi_2)}{\left\{H(\alpha,\beta,\mu,a,c,\xi_2)+1-\frac{1}{\mu}-\beta\right\}}z\right.$$

$$\left.+\frac{1}{\left\{H(\alpha,\beta,\mu,a,c,\xi_2)+1-\frac{1}{\mu}-\beta\right\}}\sum_{n=2}^{\infty}H(\alpha,\beta,\mu,a,c,\xi_2)a_n z^n\right\}$$

$$> 1 - \frac{H(\alpha,\beta,\mu,a,c,\xi_2)}{\left\{H(\alpha,\beta,\mu,a,c,\xi_2) + 1 - \frac{1}{\mu} - \beta\right\}} r$$

$$- \frac{1}{\left\{H(\alpha,\beta,\mu,a,c,\xi_2) + 1 - \frac{1}{\mu} - \beta\right\}} \sum_{n=1}^{\infty} (1 - \frac{1}{\mu} - \beta) |a_n| r$$

$$> 0, |z| = r$$

This establishes the inequality (5.5), and consequently the subordination relation (5.1) is proved. The assertion (5.2) is proved using (5.1) by changing $g(z)$ as

$$g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \tag{5.7}$$

The sharpness of the multiplying factor in (5.1) can be proved by considering the function

$$f_1(z) = z - \frac{1 - \frac{1}{\mu} - \beta}{H(\alpha,\beta,\mu,a,c,\xi_2)} z^2 \tag{5.8}$$

which belings to $S^{**}(\alpha,\beta,\mu,a,c)$
using (5.1) we conclude that

$$\frac{H(\alpha,\beta,\mu,a,c,\xi_2)}{2\left\{H(\alpha,\beta,\mu,a,c,\xi_2) + 1 - \frac{1}{\mu} - \beta\right\}} f_1(z) \prec \frac{z}{1-z}, \quad z \in U$$

from $f_1(z)$ defined by (5.7) and the fact that $\frac{z}{1-z}$ maps the unit disc onto the domain $R_e w > -\frac{1}{2}$, we infer that, $\inf_{z\in U} \left(\frac{H(\alpha,\beta,\mu,a,c,\xi_2)}{\left\{H(\alpha,\beta,\mu,a,c,\xi_2) + 1 - \frac{1}{\mu} - \beta\right\}} \right) = -\frac{1}{2}, \quad z \in U$

Thus the proof is complete.

Remark 2 : In the theorem (5.1), taking $\mu = 1$, we obtain the result of R. K. Raina [17]

Next we state the following Littlewood's subordination theorem which we use in our investigation to obtain the integral mean equality

Lemma 5.1 : If $f(z)$ and $g(z)$ are analytic in U with $f(z) \prec g(z)$, then
$\int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta \leq \int_0^{2\pi} \left|g(re^{i\theta})\right|^p d\theta$
where $0 < p < \infty, \quad z = re^{io} \quad and \quad 0 < r < 1.$

The inequality holds for $0 < r < 1$, unless f is constant or $w(z) = \alpha z, \quad |\alpha| = 1$

Applying Lemma 5.1 for the function $f(z)$ in the class $S^{**}(\alpha, \beta, \mu, a, c)$, we get the following result.

Theorem 5.2 : Let $p > 0$. If $f(z) \in S^{**}(\alpha, \beta, \mu, a, c)$ is given by (1.1) and $\{\xi_n\}_{n=2}^{\infty}$ is a non - decreasing sequence, then for ,

$$\begin{aligned} z &= re^{i\theta}(0 < r < 1) \\ \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta &\leq \int_0^{2\pi} \left| g(re^{i\theta}) \right|^p d\theta \\ where & \\ f_1(z) &= z - \frac{1 - \dfrac{1}{\mu} - \beta}{H(\alpha, \beta, \mu, a, c, \xi_2)} z^2 \end{aligned}$$

We complete this theorem with the following remark.

Remark 3 : Using Theorem (5.2), we can deduce integral mean inequalities for the class $S_s(\alpha, \beta)$ and $S_k(\alpha, \beta)$ by specializing the parameters as stated in Remark 2.

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